UNDERSTANDING 3D ROTATIONS: EULER ANGLE CONVENTIONS FOR ESTIMATION, NAVIGATION, AND CONTROL

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1. INTRODUCTION

There are several widely used conventions for representing rotations in 3D space. These include *rotation matrices*, *Euler angles*, *unit quaternions*, the *exponential map*, and the *axis-angle representation* (Diebel et al., 2006). Each of these representations offers distinct advantages depending on the application context.

Among these alternatives, Euler angles are commonly used in the estimation (Marins, Yun, Bachmann, McGhee, & Zyda, 2001), control (Mokhtari & Benallegue, 2004), and navigation of aerial systems such as quadcopters (Özaslan et al., 2017) and fixed-wing platforms, owing to their intuitive construction and ease of interpretation .

Euler angles represent a rotation in 3D as a sequence of elementary rotations about non-parallel axes. These axes correspond to the basis vectors of either the current or fixed frame depending on the chosen convention. For example, roll-pitch-yaw angles,

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widely used in the avionics literature, parameterize the orientation of an aerial platform through its relative angle about $\hat{\mathbf{x}} - \hat{\mathbf{y}} - \hat{\mathbf{z}}$ basis vectors of a fixed frame (Fig. 1) (Mellinger & Kumar, 2011).

Although Euler angles representation is easy to interpret, it is deficient in several important respects. Euler angles suffer from gimbal lock a phenomenon that occurs when two of the rotation axes align (Hemingway & O'Reilly, 2018). In such a configuration, one degree-of-freedom (DoF) is lost, making the platform instantaneously uncontrollable along the lost dimension. Secondly, composing multiple rotations using Euler angles is not possible except for a few corner cases. In general configurations, Euler angles must be converted into rotation matrices or quaternions for composition, and then converted back to Euler angles. Also there are twelve distinct conventions for composing elementary rotations which might be a source of confusion if adopted conventions are not defined clearly.

This work focuses on the use of *Euler and Tait-Bryan angles* for representing 3D rotations, with particular emphasis on clearly stating the conventions used in the literature and enabling the reader to competently interpret technical documentation and software implementations involving rotational formulations.

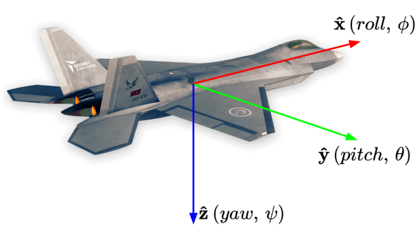


Figure 1: Roll-pitch-yaw (RPY) angles are widely used in avionics literature. This convention parametrizes the platform orientation as a sequence of elementary rotations applied in the xyz order with respect to a fixed (inertial) frame.

2. ELEMENTARY ROTATIONS

A reference frame is define through its origin and basis vectors. In Fig. 1 a reference frame, $\mathcal{B}$, is attached to an airplane body, with the basis vector $\hat{\mathbf{x}} - \hat{\mathbf{y}} - \hat{\mathbf{z}}$. In kinematic and dynamic formulations, the three basis vectors are chosen to be mutually orthogonal, *i.e.* $\hat{\mathbf{x}} \cdot \hat{\mathbf{y}} = 0, \hat{\mathbf{y}} \cdot \hat{\mathbf{z}} = 0$ and $\hat{\mathbf{x}} \cdot \hat{\mathbf{z}} = 0$, and have unit norms, *i.e.* $\hat{\mathbf{x}} \cdot \hat{\mathbf{x}} = \hat{\mathbf{y}} \cdot \hat{\mathbf{y}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{z}} = 1$. Lastly, the basis vectors form a right-handed triad, *i.e.* $\hat{\mathbf{x}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}}$, $\hat{\mathbf{y}} \times \hat{\mathbf{z}} = \hat{\mathbf{x}}$ and $\hat{\mathbf{z}} \times \hat{\mathbf{x}} = \hat{\mathbf{y}}$. Elementary rotations occur about one of the basis vectors of a given reference frame.

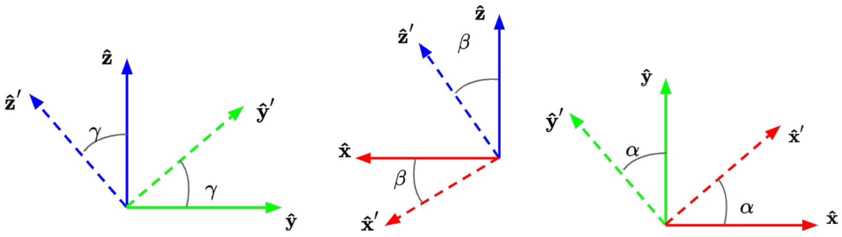
The axis about which the rotation occurs is unaffected, while the other two basis vectors rotate as shown in Fig. 2. We can obtain the respective rotation matrices through observing the coordinates of these basis vectors after rotation. For example, the vectors $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$ after rotating about $\hat{\mathbf{x}}$ by γ radians can be written, in terms of their initial values, as

$$\hat{\mathbf{x}}' = \left[1, 0, 0\right]^\top, \ \hat{\mathbf{y}}' = \left[0, c(\gamma), s(\gamma)\right]^\top, \ \hat{\mathbf{z}}' = \left[0, -s(\gamma), c(\gamma)\right]^\top. \tag{1}$$

This relation can easily be confirmed by observing Fig. 2a. Stacking these vectors side-by-side one can obtain the elementary rotation matrix about the respective axis as given below

$$\mathbf{R}_{\hat{\mathbf{x}}}(\gamma) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c(\gamma) & -s(\gamma) \\ 0 & s(\gamma) & c(\gamma) \end{bmatrix}, \quad \mathbf{R}_{\hat{\mathbf{y}}}(\beta) = \begin{bmatrix} c(\beta) & 0 & s(\beta) \\ 0 & 1 & 0 \\ -s(\beta) & 0 & c(\beta) \end{bmatrix}$$
$$\mathbf{R}_{\hat{\mathbf{z}}}(\alpha) = \begin{bmatrix} c(\alpha) & -s(\alpha) & 0 \\ s(\alpha) & c(\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2}$$

where the subscripts denote the axis of rotation.



(a) $\hat{\mathbf{x}}$-axis (b) $\hat{\mathbf{y}}$-axis (c) $\hat{\mathbf{z}}$-axis

Figure 2: The effect of elementary rotations on the other two basis vectors: from left to right, rotations about the $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ axes.

3. COMPOSING ROTATIONS

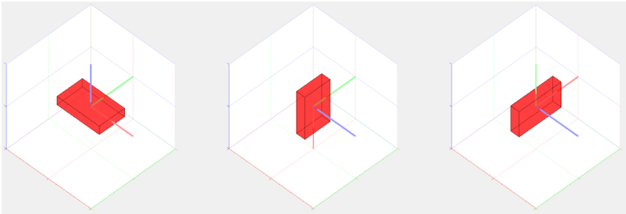
Rotations can be combined through matrix multiplication. It should be noted that the order in which the rotation matrices are multiplied is important since matrix multiplication is non-commutative except for special cases, *i.e.* $\mathbf{A} \cdot \mathbf{B} \neq \mathbf{B} \cdot \mathbf{A}$ for most $\mathbf{A}$ and $\mathbf{B}$. As an example, consider the two elementary rotations $\mathbf{R}_{\hat{\mathbf{y}}}\left(\frac{\pi}{2}\right)$ and $\mathbf{R}_{\hat{\mathbf{z}}}\left(\frac{\pi}{2}\right)$ applied at different orders on an a reference frame initially at identity orientation as shown in Fig. 3. The elementary rotation matrices are

$$\mathbf{R}_{\hat{\mathbf{y}}}\left(\frac{\pi}{2}\right) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \quad , \quad \mathbf{R}_{\hat{\mathbf{z}}}\left(\frac{\pi}{2}\right) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3}$$

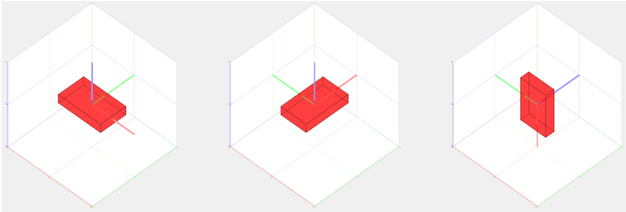
which results in the following orientations

$$\mathbf{R}_{\hat{\mathbf{y}}}\left(\pi/2\right)\mathbf{R}_{\hat{\mathbf{z}}}\left(\pi/2\right) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \mathbf{R}_{\hat{\mathbf{z}}}\left(\pi/2\right)\mathbf{R}_{\hat{\mathbf{y}}}\left(\pi/2\right) = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}. \tag{4}$$

It is evident from both geometric and algebraic perspectives that the order in which rotations are applied is critical and may lead to different final orientations.



(a) Rotations about $\hat{\mathbf{y}}$ and then $\hat{\mathbf{z}}$.



(b) Rotations about $\hat{\mathbf{z}}$ and then $\hat{\mathbf{y}}$.

Figure 3: Consecutive elementary rotations applied on a rectangular prism about $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$ at different orders to demonstrate that rotations are not commutative. It should be noted that the second rotation is applied with respect to the *current frame*.

3.1. Intrinsic and Extrinsic Rotations

In the above discussion, although the resulting orientations differ, both compositions produce valid rotation matrices. This underscores a concept of fundamental importance in rotational kinematics: the outcome of successive rotations depends critically on their order. Equally important is the *reference frame* with respect to which each rotation is applied. Specifically, a rotation can be performed either relative to the *fixed (inertial) frame*, referred to as an *extrinsic rotation*, or relative to the *current (rotating) frame*, which is known as an *intrinsic rotation*. This distinction leads to different interpretations and implementations, particularly in applications such as robotics, aerospace, and computer graphics.

In the case of *intrinsic rotations*, where each new rotation is applied relative to the current (rotating) frame, composition is performed by multiplying the existing rotation matrix from the *right* by the new rotation. Conversely, for *extrinsic rotations*, where rotations are applied relative to the fixed (inertial) frame, composition is performed by multiplying the new rotation from the *left*.

Proof: Consider a rotation $\mathbf{R}_i$ applied to an initial identity orientation, where i denotes the axis of rotation aligned with a basis vector of the fixed (initial) frame. To apply a new rotation, $\mathbf{R}_j$ where j is a basis vector, with respect to the *fixed frame*, we must virtually align the current (rotated) frame with the fixed frame before applying the new rotation.

This alignment is achieved by *undoing* the current orientation, which can be performed by right-multiplying the current rotation matrix by its inverse, $\mathbf{R}_i^{-1}$. Although this may seem trivial, it allows us to apply the new elementary rotation matrix, $\mathbf{R}_j$, with respect to the fixed frame (which, at that moment, is effectively the current frame).

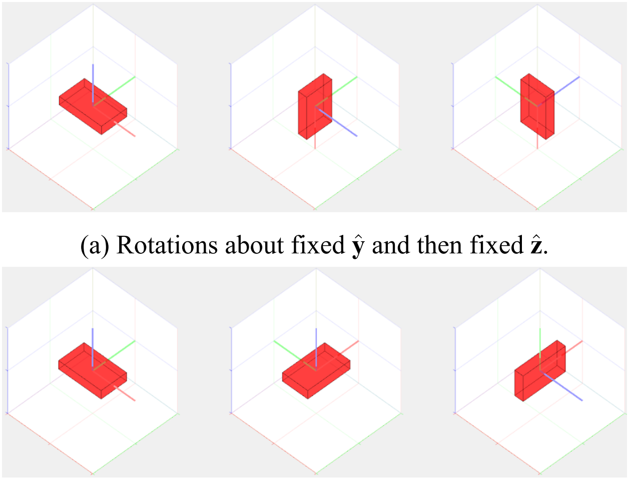
After applying the new rotation, $\mathbf{R}_j$, we reapply the original orientation, $\mathbf{R}_i$, by right-multiplying the result with the original rotation matrix. In total, this composition corresponds to *left-multiplication* of the new rotation matrix onto the current orientation, thereby validating the composition rule for extrinsic rotations. These steps are summarized in Table 1

In order to compare the effect of order of rotations, we will again consider the two elementary rotations $\mathbf{R}_{\hat{\mathbf{y}}}\left(\frac{\pi}{2}\right)$ and $\mathbf{R}_{\hat{\mathbf{z}}}\left(\frac{\pi}{2}\right)$ applied at different orders, but applied with respect to the fixed frame. The resultant rotation matrices write

$$\mathbf{R}_{\hat{\mathbf{y}}}\left(\pi/2\right)\mathbf{R}_{\hat{\mathbf{z}}}\left(\pi/2\right) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \mathbf{R}_{\hat{\mathbf{z}}}\left(\pi/2\right)\mathbf{R}_{\hat{\mathbf{y}}}\left(\pi/2\right) = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}. \tag{5}$$

Step	Current Orientation	Explanation
1	$\mathbf{R}^{(1)} = \mathbf{R}_i$	Initial rotation about axis i.
2	$\mathbf{R}^{(2)} = \mathbf{R}_i \cdot \mathbf{R}_i^{-1}$	Undo the effect of the initial rotation. This temporarily aligns the current frame with the fixed (initial) frame.
3	$\mathbf{R}^{(3)} = \mathbf{R}_i \cdot \mathbf{R}_i^{-1} \cdot \mathbf{R}_j$	Apply a new rotation about axis j, one of the axes of the fixed (initial) frame.
4	$\mathbf{R}^{(4)} = \left(\mathbf{R}_i \cdot \mathbf{R}_i^{-1}\right) \cdot \mathbf{R}_j \cdot \mathbf{R}_i$	Reapply the undone rotation to restore the frame to its original orientation before the new rotation.
5	$\mathbf{R}^{(4)} = \mathbf{R}_j \mathbf{R}_i$	Since $\mathbf{R}_i \cdot \mathbf{R}_i^{-1} = \mathbf{I}$, we simplify to obtain the standard extrinsic rotation composition.

Table 1: Steps illustrating extrinsic rotation using the logic of undoing and reapplying previously applied rotations. Orientation at each step is denoted by $\mathbf{R}^{(n)}$ where n is the proof step.



(a) Rotations about fixed $\hat{\mathbf{y}}$ and then fixed $\hat{\mathbf{z}}$.

(b) Rotations about fixed $\hat{\mathbf{z}}$ and then fixed $\hat{\mathbf{y}}$.

Figure 4: Consecutive elementary rotations applied about fixed $\hat{\mathbf{y}}$ and fixed $\hat{\mathbf{z}}$ axes demonstrating *extrinsic rotation*.

The first equations is obtained by first applying a rotation about fixed-$\hat{\mathbf{z}}$ and then about fixed-$\hat{\mathbf{y}}$; and the second one rotating about the fixed-$\hat{\mathbf{y}}$ and then fixed-$\hat{\mathbf{z}}$. As these examples illustrate, applying rotations in reverse order under the opposite convention yields the same final orientation.

Remark 1. Applying a sequence of rotations *all* in the intrinsic convention is equivalent to applying the same sequence in reverse order using the extrinsic convention, and vice versa. This equivalence yields the same final orientation despite the differing frame of reference.

Example: Now let's work out a more comprehensive example involving five sequential elementary rotations, mixing intrinsic (current frame) and extrinsic (fixed frame) conventions. The superscript on each rotation axis indicates the index of the intermediate frame with respect to which the rotation is applied, and the subscript indicates the rotation axis. The initial orientation is denoted as $\mathbf{R}^{(0)} = \mathbf{I}$ and omitted in the following calculations since the identity rotation has no effect.

Step 1: Rotate by α about the current $\hat{\mathbf{x}}$-axis:

$$\begin{aligned}\mathbf{R}^{(1)} &= \mathbf{R}^{(0)} \cdot \mathbf{R}_{\hat{\mathbf{x}}^{(0)}}(\alpha) \\ &= \mathbf{R}_{\hat{\mathbf{x}}^{(0)}}(\alpha)\end{aligned} \tag{6}$$

It should be noted that $\hat{\mathbf{x}}^{(0)} = \hat{\mathbf{x}}^{(1)}$ since the axis of rotation is an eigenvector with identity eigenvalue of the respective rotation matrix, thus is not affected. In the subsequent steps, we retain the index from the previous step as a matter of notational consistency, even though the axis itself does not change.

Step 2: Rotate by γ about the current $\hat{\mathbf{y}}$-axis (intrinsic rotation):

$$\begin{aligned}\mathbf{R}^{(2)} &= \mathbf{R}^{(1)} \cdot \mathbf{R}_{\hat{\mathbf{y}}^{(1)}}(\gamma) \\ &= \mathbf{R}_{\hat{\mathbf{x}}^{(0)}}(\alpha) \cdot \mathbf{R}_{\hat{\mathbf{y}}^{(1)}}(\gamma)\end{aligned} \tag{7}$$

which is equivalent to writing

$$\begin{aligned}\mathbf{R}^{(2)} &= \mathbf{R}_{\hat{\mathbf{x}}^{(1)}}(\alpha) \cdot \mathbf{R}_{\hat{\mathbf{y}}^{(1)}}(\gamma) \quad \text{and} \\ &= \mathbf{R}_{\hat{\mathbf{x}}^{(1)}}(\alpha) \cdot \mathbf{R}_{\hat{\mathbf{y}}^{(2)}}(\gamma)\end{aligned} \tag{8}$$

due to the discussion in the previous step.

Step 3: Rotate by θ about the fixed $\hat{\mathbf{z}}$-axis (extrinsic rotation): We first undo the effect of the previously applied rotations, through right-multiplication by $\left(\mathbf{R}^{(2)}\right)^{-1}$, apply the new elementary rotation about the fixed $\hat{\mathbf{z}}$-axis, and then reapply the undone transformations, $\mathbf{R}^{(2)}$. This process is formalized as follows:

$$\begin{aligned}\mathbf{R}^{(3)} &= \mathbf{R}^{(2)} \cdot \left[\left(\mathbf{R}^{(2)}\right)^{-1} \cdot \mathbf{R}_{\hat{\mathbf{z}}^{(0)}}(\theta) \cdot \left(\mathbf{R}^{(2)}\right)\right] \\ &= \mathbf{R}_{\hat{\mathbf{z}}^{(0)}}(\theta) \cdot \left(\mathbf{R}^{(2)}\right) \\ &= \mathbf{R}_{\hat{\mathbf{z}}^{(0)}}(\theta) \cdot \mathbf{R}_{\hat{\mathbf{x}}^{(0)}}(\alpha) \cdot \mathbf{R}_{\hat{\mathbf{y}}^{(1)}}(\gamma)\end{aligned} \tag{9}$$

Observe that the first pair of matrices evaluated to identity and are excluded. Thus, we conclude that applying a rotation about a fixed axis is equivalent to pre-multiplying the corresponding elementary rotation matrix with the current orientation.

Step 4: Rotate by β about the current $\hat{\mathbf{y}}$-axis (intrinsic rotation):

$$\begin{aligned}\mathbf{R}^{(4)} &= \mathbf{R}^{(3)} \cdot \mathbf{R}_{\hat{\mathbf{y}}^{(3)}}(\beta) \\ &= \mathbf{R}_{\hat{\mathbf{z}}^{(0)}}(\theta) \cdot \mathbf{R}_{\hat{\mathbf{x}}^{(0)}}(\alpha) \cdot \mathbf{R}_{\hat{\mathbf{y}}^{(1)}}(\gamma) \cdot \mathbf{R}_{\hat{\mathbf{y}}^{(3)}}(\beta)\end{aligned} \tag{10}$$

Step 5: Rotate by δ about the fixed $\hat{\mathbf{x}}$-axis (intrinsic rotation):

$$\begin{aligned}\mathbf{R}^{(5)} &= \mathbf{R}^{(4)} \cdot \left[\left(\mathbf{R}^{(4)}\right)^{-1} \cdot \mathbf{R}_{\hat{\mathbf{x}}^{(0)}}(\delta) \cdot \left(\mathbf{R}^{(4)}\right)\right] \\ &= \mathbf{R}_{\hat{\mathbf{x}}^{(0)}}(\delta) \cdot \mathbf{R}_{\hat{\mathbf{z}}^{(0)}}(\theta) \cdot \mathbf{R}_{\hat{\mathbf{x}}^{(0)}}(\alpha) \cdot \mathbf{R}_{\hat{\mathbf{y}}^{(1)}}(\gamma) \cdot \mathbf{R}_{\hat{\mathbf{y}}^{(3)}}(\beta)\end{aligned} \tag{11}$$

The resultant rotation matrix in Eqn. 11 could have been obtained through other rotation sequences. One possible such sequence is composition of intrinsic elementary rotations about the axes and angles as they appear in Eqn. 11. That is, rotation about $\hat{\mathbf{x}}$ by δ, then $\hat{\mathbf{z}}$ by θ, then $\hat{\mathbf{x}}$ by α, then $\hat{\mathbf{y}}$ by γ and finally about $\hat{\mathbf{y}}$ by β radians, all with respect to the current (rotating) frame gives the same orientation. These observations will be useful when we discuss different Euler angle conventions in the subsequent sections.

4. EULER AND TAIT-BRYAN ANGLES

Leonhard Euler, in his seminal work in 1776, states that orientations of rigid bodies in three dimensional space can be represented using successively applied three elementary rotations about coordinate axes (basis vectors) (Pio, 1966). Euler primarily studied the zxz sequence where the angles about each axes are called yaw (ϕ), nutation (θ) and precession (ψ). Structures with the first and the last axes being equal are commonly referred to as a *proper Euler angle sequence*, and the elementary rotations are applied with respect to the current frame unless otherwise is explicitly stated. Proper Euler angle are widely used in application areas such as classical and celestial mechanics, and analytical dynamics.

Later in the late 19^{th} and early 20^{th} centuries, researchers Peter Guthrie Tait and George Hartley Bryan proposed variations with distinct axes of rotations. Tait and Bryan extensively studied the xyz sequence which is now referred to as *roll-pitch-yaw* angles. Angles are usually represented with symbols $\phi - \theta - \psi$. Sequences with distinct axes of rotations became particularly useful in the emerging fields of aviation and navigation (Fig. 1).

4.1. Possible Sequences and Conventions

There are a total of $3^3 = 27$ possible axis sequences of three rotations, such as xyx, xyz, xzx, and xzy. However, only twelve of these constitute valid Euler or Tait-Bryan angle sequences. The remaining fifteen sequences involve repeated adjacent axes, such as xxx, xxy, or xyy, and are not used in practice due to lack of independent rotational degrees of freedom hence such sequences have either 1 or 2 DoF. Among the twelve valid configurations, six are classified as *proper Euler angle* sequences , and the other six as *Tait-Bryan angle* sequences. The former class has the first and third rotation axes the same and the latter have all three rotation axes distinct (Table 2).

Each class of sequences, Euler and Tait-Bryan, can be formulated using either intrinsic or extrinsic composition conventions. However due to various reasons such as terminological consistency, conventions used in the literature and software packages and interpretability, intrinsic composition convention is preferred unless otherwise is clearly stated. As explained in the previous sections, it is always possible to express any given sequence in the reverse order thereby switching from intrinsic to extrinsic convention, and vice versa, when required. In summary, while an extrinsic interpretation of Euler and Tait-Bryan angles can be defined consistently from a mathematical standpoint, its use should be stated clearly avoid misinterpretation.

Irrespective of the conventions used, only three angles along with the axes of rotations, suffice to represent any rotation in $\mathcal{SO}(3)$ making Euler/Tait-Bryan angles minimal in terms of representation.

Category	**Axis Sequences**
Improper Sequences	xxx, xxy, xxz, yyx, yyy, yyz, zzx, zzy, zzz, xyy, yxx, yzz, zyy, xxz
Proper Euler Sequences	zxz, xyx, yzy, zyz, xzx, yxy
Tait–Bryan Sequences	xyz, xzy, yxz, yzx, zxy, zyx

Table 2: Categorization of 3-element axis sequences into improper, proper Euler, and Tait-Bryan categories.

4.2. zyz Euler Angle Sequence:

In this section we will derive the rotation matrix for the zyz Euler angle sequence. Embracing the general convention we apply the elementary rotations with respect to the current frame and the angles represented as yaw (ϕ), nutation (θ) and precession (ψ). The

```cpp
double phi = ..., theta = ..., psi = ...;

Eigen::AngleAxisd Rz1(phi, Eigen::Vector3d::UnitZ());
Eigen::AngleAxisd Ry(theta, Eigen::Vector3d::UnitY());
Eigen::AngleAxisd Rz2(psi, Eigen::Vector3d::UnitZ());

Eigen::Matrix3d R = Rz1 * Ry * Rz2;
```

Listing 1: `c++` code snippet for ZYZ intrinsic rotation using `Eigen` library

elementary rotation matrices are written as

$$\mathbf{R}_{\hat{\mathbf{z}}^{(0)}}(\phi) = \begin{bmatrix} c(\phi) & -s(\phi) & 0 \\ s(\phi) & c(\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad , \quad \mathbf{R}_{\hat{\mathbf{y}}^{(1)}}(\theta) = \begin{bmatrix} c(\theta) & 0 & s(\theta) \\ 0 & 1 & 0 \\ -s(\theta) & 0 & c(\theta) \end{bmatrix}$$

$$\mathbf{R}_{\hat{\mathbf{z}}^{(2)}}(\psi) = \begin{bmatrix} c(\psi) & -s(\psi) & 0 \\ s(\psi) & c(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix} . \tag{12}$$

The resultant rotation matrix is obtained as

$$\mathbf{R}_{\mathtt{zyz}}(\phi, \theta, \psi) = \mathbf{R}_{\hat{\mathbf{z}}^{(0)}}(\phi) \cdot \mathbf{R}_{\hat{\mathbf{y}}^{(1)}}(\theta) \cdot \mathbf{R}_{\hat{\mathbf{z}}^{(2)}}(\psi) \tag{13}$$

$$= \begin{bmatrix} c(\phi)c(\theta)c(\psi) - s(\phi)s(\psi) & -c(\phi)c(\theta)s(\psi) - s(\phi)c(\psi) & c(\phi)s(\theta) \\ s(\phi)c(\theta)c(\psi) + c(\phi)s(\psi) & -s(\phi)c(\theta)s(\psi) + c(\phi)c(\psi) & s(\phi)s(\theta) \\ -s(\theta)c(\psi) & s(\theta)s(\psi) & c(\theta) \end{bmatrix}$$

The same resultant matrix could have been obtain by using the `ZYZ` sequence with extrinsic convention and angles ψ, θ, ϕ which would write

$$\mathbf{R}^{(ext)}_{\mathtt{zyz}}(\psi, \phi, \theta) = \mathbf{R}_{\hat{\mathbf{z}}^{(0)}}(\phi) \cdot \mathbf{R}_{\hat{\mathbf{y}}^{(0)}}(\theta) \cdot \mathbf{R}_{\hat{\mathbf{z}}^{(0)}}(\psi) \tag{14}$$

We provide a `c++` code snippet in `Eigen` (Guennebaud, Jacob, et al., n.d.) for this sequence with intrinsic convention in Listing 1.

4.2.1. Solving for zyz Euler Angles:

We have shown how to obtain the rotation matrix given the Euler angles for the zyz sequence in Equation 13. The inverse problem, however, requires extracting the angles from a given rotation matrix. Upon inspecting the structure of Equation 13, it becomes evident that the top-left 2×2 submatrix is heavily coupled, making it unsuitable for direct analytical inversion. In contrast, other elements of the matrix exhibit clearer patterns that can be exploited to solve for the individual Euler angles.

Before we proceed further, we enumerate the elements of the matrix as

$$\mathbf{R} = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix}. \tag{15}$$

Next we observe the elements R_{11}, R_{21} and R_{31}, R_{32} all of which include $s(\theta)$. Thus, for $\theta = \frac{(2n+1)\pi}{2}$ for $n \in \mathbb{Z}$ all of these terms become 0 which is a special case. Depending on the sign of $s(\theta)$, we can determine the angles using different formula. If $s(\theta) > 0$, or equivalently $\theta \in (0, \pi)$, the solution take the following form

$$\phi = \text{atan2}(R_{23}, R_{13}) \quad , \quad \psi = \text{atan2}(R_{32}, -R_{31})$$
$$\theta = \text{atan2}\left(\sqrt{R_{13}^2 + R_{23}^2}, R_{33}\right). \tag{16}$$

On the other hand if $s(\theta) < 0$, or equivalently $\theta \in (\pi, 2\pi)$, we get

$$\phi = \text{atan2}(-R_{23}, -R_{13}) \quad , \quad \psi = \text{atan2}(-R_{32}, R_{31})$$
$$\theta = \text{atan2}\left(-\sqrt{R_{13}^2 + R_{23}^2}, R_{33}\right) \tag{17}$$

In the special case that $s(\theta) = 0$, either $\theta = 2n\pi$ or $\theta = (2n+1)\pi$

for $n \in \mathbb{Z}$. In the former case Eqn. 13 takes the following form

$$\begin{aligned}
&\mathbf{R}_{\mathtt{zyz}}(\phi, 2n\pi, \psi) \\
&= \begin{bmatrix} c(\phi)c(\psi) - s(\phi)s(\psi) & -c(\phi)s(\psi) - s(\phi)c(\psi) & 0 \\ s(\phi)c(\psi) + c(\phi)s(\psi) & -s(\phi)s(\psi) + c(\phi)c(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \cos(\phi+\psi) & -\sin(\phi+\psi) & 0 \\ \sin(\phi+\psi) & \cos(\phi+\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned} \tag{18}$$

and the latter case gives

$$\begin{aligned}
&\mathbf{R}_{\mathtt{zyz}}\left(\phi, (2n+1)\pi, \psi\right) \\
&= \begin{bmatrix} -c(\phi)c(\psi) - s(\phi)s(\psi) & c(\phi)s(\psi) - s(\phi)c(\psi) & 0 \\ -s(\phi)c(\psi) + c(\phi)s(\psi) & s(\phi)s(\psi) + c(\phi)c(\psi) & 0 \\ 0 & 0 & -1 \end{bmatrix} \\
&= \begin{bmatrix} -\cos(\phi-\psi) & -\sin(\phi-\psi) & 0 \\ -\sin(\phi-\psi) & \cos(\phi-\psi) & 0 \\ 0 & 0 & -1 \end{bmatrix}.
\end{aligned} \tag{19}$$

It clearly observed in the special case $s(\theta) = 0$, one degree of freedom is lost since for a given rotation matrix, one can only solve for the two quantities, θ and $\phi \pm \psi$. Individual values of ϕ and ψ cannot be determined and this is called a *gimbal lock*.

4.2.2. Gimbal Lock for zyz Euler Sequence

A *gimbal lock* occurs when two of the rotation axes align. In the case of zyz Euler sequence, this happened when the first and the last z axes align. We have shown in the previous section that $\theta = 2n\pi$ and $\theta = (2n+1)\pi$ result in two different gimbal locks. In the former case the intermediate rotation never happens since $\mathbf{R}^{(1)}_{\hat{\mathbf{y}}}(2n\pi) = \mathbf{I}$ causing $\hat{\mathbf{z}}$ to remain unrotated, *i.e.* $\hat{\mathbf{z}}^{(0)} = \hat{\mathbf{z}}^{(2)}$. In the latter case $\mathbf{R}_{\hat{\mathbf{y}}}^{(1)}(2n\pi)$ results in $\hat{\mathbf{z}}^{(0)} = -\hat{\mathbf{z}}^{(2)}$, *i.e.* the first and the last axes of rotations become anti-parallel. In both cases only one rotation about the same (or anti-parallel) axis.

5. EULER ANGLE KINEMATICS

In a typical attitude estimation scenario which employs an Inertial Measurement Unit (IMU), an onboard gyroscope provides instantaneous angular velocity measurements which can be integrated over time to estimate the platform orientation. While there is huge literature on attitude estimation using accelerometer, gyroscope and magnetometer which are usually packed into a single IMU, the theoretical fundamentals and vast technical nuances are beyond the scope of the this work. However, we present the relation between angular rates measured in sensor/body frame ($\mathcal{B}$) and the Euler angles. We will consider the $\mathtt{zyz}$ sequence to study this.

In order to obtain such a relation we need to write each of the intermediate rotation axes with respect to the sensor/body frame, $\mathcal{B}$. We will use the notation introduced in the rest of the work and enumerate axes with the index of the intermediate reference frames. For example, the first, second and last rotations occur about $\hat{\mathbf{z}}^{(0)}$, $\hat{\mathbf{y}}^{(1)}$ and $\hat{\mathbf{z}}^{(2)}$. Since these are the axes of rotation of the respective step, they, respectively, are equal to $\hat{\mathbf{z}}^{(1)}$, $\hat{\mathbf{y}}^{(2)}$ and $\hat{\mathbf{z}}^{(3)}$. Let's express each of these with respect to the body frame basis vectors which are $\hat{\mathbf{x}}^{(3)}$, $\hat{\mathbf{y}}^{(3)}$ and $\hat{\mathbf{z}}^{(3)}$ as the final frame is by definition the body frame, $\mathcal{B}$. The basis vectors at the intermediate steps 1, 2 and 3 are related as

$$\begin{bmatrix} \hat{\mathbf{x}}^{(2)} \\ \hat{\mathbf{y}}^{(2)} \\ \hat{\mathbf{z}}^{(2)} \end{bmatrix} = \begin{bmatrix} c(\psi) & -s(\psi) & 0 \\ s(\psi) & c(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix}^{\top} \begin{bmatrix} \hat{\mathbf{x}}^{(3)} \\ \hat{\mathbf{y}}^{(3)} \\ \hat{\mathbf{z}}^{(3)} \end{bmatrix} \tag{20}$$

$$\begin{bmatrix} \hat{\mathbf{x}}^{(1)} \\ \hat{\mathbf{y}}^{(1)} \\ \hat{\mathbf{z}}^{(1)} \end{bmatrix} = \begin{bmatrix} c(\theta) & 0 & s(\theta) \\ 0 & 1 & 0 \\ -s(\theta) & 0 & c(\theta) \end{bmatrix}^\top \begin{bmatrix} \hat{\mathbf{x}}^{(2)} \\ \hat{\mathbf{y}}^{(2)} \\ \hat{\mathbf{z}}^{(2)} \end{bmatrix} \tag{21}$$

where the rotation matrices, respectively, are $\mathbf{R}^\top_{\hat{\mathbf{z}}^{(2)}}$ and $\mathbf{R}^\top_{\hat{\mathbf{y}}^{(1)}}$. The body angular velocity in terms of the Euler angular rates is

$$\omega = \dot{\phi}\hat{\mathbf{z}}^{(1)} + \dot{\theta}\hat{\mathbf{y}}^{(2)} + \dot{\psi}\hat{\mathbf{z}}^{(3)}. \tag{22}$$

At this point we need to express all vectors in the above expression in terms of basis vectors of the final, 3^{rd}, frame as

$$\hat{\mathbf{z}}^{(1)} = s(\theta)\hat{\mathbf{x}}^{(2)} + c(\theta)\hat{\mathbf{z}}^{(2)} \tag{23}$$

$$\hat{\mathbf{y}}^{(2)} = -s(\psi)\hat{\mathbf{x}}^{(3)} + c(\psi)\hat{\mathbf{y}}^{(3)}. \tag{24}$$

But this requires us to write the following relations as well

$$\hat{\mathbf{x}}^{(2)} = c(\psi)\hat{\mathbf{x}}^{(3)} + s(\psi)\hat{\mathbf{y}}^{(3)} \tag{25}$$

$$\hat{\mathbf{z}}^{(2)} = \hat{\mathbf{z}}^{(3)} \tag{26}$$

which gives

$$\begin{aligned} \hat{\mathbf{z}}^{(1)} &= s(\theta)\Big(c(\psi)\hat{\mathbf{x}}^{(3)} + s(\psi)\hat{\mathbf{y}}^{(3)}\Big) + c(\theta)\hat{\mathbf{z}}^{(3)} \\ &= s(\theta)c(\psi)\hat{\mathbf{x}}^{(3)} + s(\theta)s(\psi)\hat{\mathbf{y}}^{(3)} + c(\theta)\hat{\mathbf{z}}^{(3)}. \end{aligned} \tag{27}$$

Thus the angular velocity in Euler angle rates can be written as

$$\begin{aligned} \omega = \dot{\phi}&\Big(s(\theta)c(\psi)\hat{\mathbf{x}}^{(3)} + s(\theta)s(\psi)\hat{\mathbf{y}}^{(3)} + c(\theta)\hat{\mathbf{z}}^{(3)}\Big) \\ &+ \dot{\theta}\Big(- s(\psi)\hat{\mathbf{x}}^{(3)} + c(\psi)\hat{\mathbf{y}}^{(3)}\Big) + \dot{\psi}\hat{\mathbf{z}}^{(3)} \end{aligned} \tag{28}$$

$$\begin{aligned} &= \hat{\mathbf{x}}^{(3)}\Big(\dot{\phi}s(\theta)c(\psi) - \dot{\theta}s(\psi)\Big) + \\ &\hat{\mathbf{y}}^{(3)}\Big(\dot{\phi}s(\theta)s(\psi) + \dot{\theta}c(\psi)\Big) + \hat{\mathbf{z}}^{(3)}\Big(\dot{\phi}\, c(\theta) + \dot{\psi}\Big). \end{aligned} \tag{29}$$

In the robotics literature the angular rates as measured by a gyroscope are usually denoted $[p, q, r]^\top$ and defined in the 3^{rd} frame (also $\mathcal{B}$). Hence these and the Euler angular rates are now defined in the same frame and can be related as

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} s(\theta)c(\psi) & -s(\psi) & 0 \\ s(\theta)s(\psi) & c(\psi) & 0 \\ c(\theta) & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}. \tag{30}$$

Having this relation established, gyroscope measurements can be transformed into Euler angular rates which then can be numerically integrated to directly update the orientation estimate in Euler domain.

6. CONCLUSION

Euler angles are widely used in the robotics and computer vision communities as a means of representing orientation in three-dimensional space. Their appeal lies in their intuitive geometric interpretation and ease of visualization. However, for algorithmic and computational purposes, Euler angles are often converted into more tractable representations such as rotation matrices or unit quaternions.

In this work, we focused on the zyz Euler angle convention. We derived the associated rotation matrix, examined the inverse problem of recovering Euler angles from a given rotation matrix, and analyzed singular configurations and corner cases that arise in practical applications. Furthermore, we established a relationship between angular velocity, typically measured by a gyroscope, and the time derivatives of the zyz Euler angles. This derivation provides a useful bridge between sensor measurements and orientation parameters.

The methodology and illustrative examples presented in this work can be extended to other Euler sequences. We hope this work serves as a foundational reference for researchers and practitioners seeking to understand and employ Euler angle conventions in 3D orientation estimation tasks.

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